

Identifying the Bose glass phase

R. Pugatch, N. Bar-Gill, N. Katz, E. Rowen, and N. Davidson
*Department of Physics of Complex Systems,
 Weizmann Institute of Science, Rehovot 76100, Israel*

Introducing disorder into the Bose-Hubbard model at integer fillings leads to a Bose glass phase, along with the Mott insulator and superfluid phases [1]. We suggest a new order parameter: $\det(\rho)$, where ρ is the one body density matrix, which is nonzero only within the Mott-insulator phase. Alongside the superfluid fraction f_s , it is then possible to distinguish the three phases. The Bose glass phase is the only phase which has vanishing $\det(\rho)$ and superfluid fraction. The vanishing of $\det \rho$ in the Bose glass phase occurs due to the partial fragmentation of the condensate into localized fragments, each with zero superfluid response, which implies the presence of unoccupied sites and hence the presence of lines of zeros in the one body density matrix. In the superfluid phase, $\det \rho$ vanishes for another reason - due to the macroscopic occupation of a single particle state. Finally, we suggest the enhancement of the three body decay rate in the Bose glass phase, as an experimental indicator for the presence of localized fragments.

I. INTRODUCTION

The onset of superfluidity in random media and its connection to the existence of a condensate fraction has been an intriguing research subject in the ^4He community for many years [2, 3]. In recent years, ultra-cold atomic states in optical lattices have provided an experimental means to answering these long-standing problems [4].

The remarkable tunability and dynamical control of optical potentials offers the opportunity to emulate elaborate physical situations and create novel quantum many-body states of fundamental interest. In particular, the study of disordered systems comprises an important avenue of research. Specifically, the Anderson localization transition, perhaps the best known example for a quantum phase transition from an extended to a localized state, has never been unambiguously observed in an experiment. Interacting bosons in a disordered potential can present interesting and sometimes unexpected behavior, as we shall discuss shortly. It is known [1, 5] that interacting lattice bosons in the presence of a disordered on-site potential exhibit a phase known as Bose glass, characterized as an insulating phase with no gap in the excitation spectrum.

The Bose-Hubbard Hamiltonian [5] with on-site disorder is given by

$$H = -\frac{J}{2} \sum_i \hat{a}_i^\dagger \hat{a}_{i+1} + h.c. + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i + \sum_i W_i \hat{a}_i^\dagger \hat{a}_i, \quad (1)$$

where a_i is the annihilation operator for an atom at site i , J is the tunnelling strength, U is the repulsive on site interaction strength, and W_i is the random on site potential, drawn from a flat distribution centered around zero, of width Δ .

When $W_i = 0$ for all i , this Hamiltonian exhibits a second order quantum phase transition from an energy gapped Mott-insulator to a gapless superfluid state at integer filling (number of atoms equal the number of sites), as the ratio U/J is changed [5]. The critical value for this transition depends on the dimensionality of the underlying

lattice.

This model was extensively studied by a variety of methods, including quantum monte-carlo in 1D and 2D [6, 7], exact solutions in 1D for small lattices [8], renormalization group [9], and recently in the context of cold atoms in optical lattices using the mean field approximation [10]. One of the main conclusions of these studies was that due to the disorder, there is a new incompressible yet insulating phase, named the Bose glass phase. It was conjectured by Fisher et. al. [1], and verified by [11] and [12] that any transition from the Mott-insulator phase to the superfluid phase in the presence of disorder, should follow through the Bose glass phase (at integer filling).

In this paper we suggest a new criterion for identifying the Bose glass phase, based on the comparison between the superfluid fraction, and the structure of the one body density function. We show that in the presence of interactions, however small, a localized condensate becomes unstable, and fragmentation occurs. In the thermodynamic limit and a finite interaction strength, the number of localized eigenfunctions scales with the size of the system, so the occupation of each fragment (eigenfunction of the one body density matrix) tends to zero. As we will explain below, one can use two order parameters, the superfluid fraction f_s , and $\det \rho$, where ρ is the one body density matrix, to distinguish between the three phases, since f_s is non-zero only in the superfluid phase and $\det(\rho)$ is non-zero only in the Mott insulator phase (where $f_s = 0$).

II. INTERPLAY BETWEEN KINETIC, POTENTIAL AND INTERACTION ENERGY

In this section we will use a variational argument, to demonstrate how the relative strength of the kinetic potential and interaction energies, can stabilize a different ground state, either a Mott insulator (strong interaction limit), Superfluid (both disorder and interaction comparable to the kinetic energy), and Bose glass phase (dis-

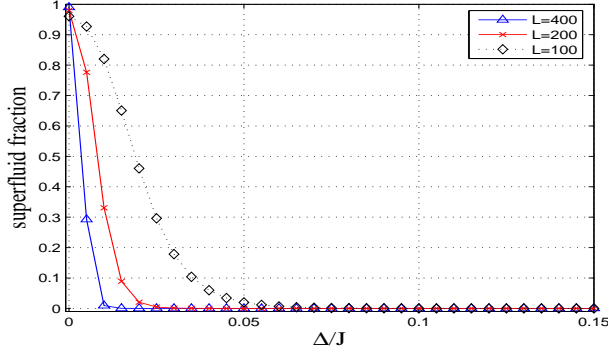


FIG. 1: Superfluidity as a function of the disorder strength for different lattice sizes at zero interaction energy. Note that this superfluid fraction is the response of the many-body wave function to a phase twist in the boundary conditions, hence it is equal to 100% for zero disorder and interaction strength, without contradicting the Landau definition, which is a stability criterion [15].

order or interaction or both are dominant). Surprisingly, we will find regimes where by either increasing the interaction strength U or the disorder strength Δ we can induce superfluidity in the system. The latter phenomenon deserves a special name and is called disorder induced order (see e.g. [13, 14]). In what follows, we devise a variational argument that compares three typical limiting forms of the many body ground state, and show how as a function of U and Δ the nature of the ground state changes in accord with the known phase diagram of the system. We further point out how this result supports our suggested criteria.

Consider first, a 1D, non interacting, single particle disordered system with the following Hamiltonian:

$$H = -\frac{J}{2}(\sum_i \hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_i \hat{a}_{i+1}^\dagger) + W_i \hat{a}_i^\dagger \hat{a}_i. \quad (2)$$

As before, W_i is drawn randomly from a flat distribution of width Δ centered around zero. All the eigenstates ϕ_i of this Hamiltonian are localized, and we denote the lowest energy as E_0 and the corresponding eigenstate to be ϕ_0 , so $H\phi_0 = E_0\phi_0$. Inserting into this system N non interacting bosons, all of them condense in the lowest energy state. Trivially, the total many body wave function is then given by: $\psi_{AG}(x_1, \dots, x_N) = \prod_i \phi_0(x_i)$, where AG stands for Anderson glass. This state is an example of a completely localized state, with zero superfluid response, but nevertheless, it is macroscopically occupied. Some authors refer to this state, as an Anderson glass [16, 17].

In Fig. 1, we plot the superfluid fraction of this ground state, as a function of the disorder strength for $L = 100, 200$ and 400 sites. We calculated the superfluid fraction f_s exactly using the following formula [18]

$$f_s = -\frac{1}{NJ} \left(-\frac{1}{2} \langle \phi_0 | \hat{\mathbf{T}} | \phi_0 \rangle - \sum_{i \neq 0} \frac{\langle \phi_i | \hat{\mathbf{J}} | \phi_0 \rangle}{E_i - E_0} \right), \quad (3)$$

where $\hat{\mathbf{T}}$ is the kinetic energy operator and $\hat{\mathbf{J}}$ is the current operator. In the following section we elaborate further on this definition of superfluidity.

It is evident from Fig. 1 that the critical point Δ_c where superfluidity disappears tends to $\Delta_c = 0$ in the thermodynamic limit $L \rightarrow \infty$. This is of course expected, since any disorder is strongly localizing in 1D [19]. What happens if we now turn on the interaction to a small, positive value? Assuming the usual S-wave interaction, the interaction energy of the localized state ϕ_0 is $U_0 N(N-1)/2$, where $U_0 = g \int dx |\phi_0|^4$, and g is the effective 1D interaction coefficient [20]. This energy is much larger than the energy of a fragmented state of the form $\psi_{BG} = \prod_i \phi_i(x_i)^{n_i}$ where the integers n_i are zero above some $k \leq N$, and $\sum_i n_i = N$ (as we will see shortly, the case $k = N$ corresponds to the Mott insulator phase), and $\phi_i(x)$ are the eigenstates of the free particle Hamiltonian in Eq. 2. The interaction energy of this state is approximately given by $\sum_i U_i n_i(n_i-1)/2 + \sum_{i \neq j} U_{ij} n_i n_j$, where $U_i = g \int dx |\phi_i|^4$ is the inter state interaction energy, and $U_{ij} = g \int dx |\phi_i|^2 |\phi_j|^2$ is the intra state interaction energy [21]. It is not difficult to find a set of n_i 's which gives a smaller energy than a macroscopically occupied condensate wave function of the form ψ_{AG} . To do so, we start by putting the first particle in the lowest energy state ϕ_0 we then add the next particle, searching among all states for the state that will minimize the total energy (including interaction). We repeat this process until we fill the lattice with $N = L$ particles, to obtain a many-body state that has the following form:

$$\psi_{BG}(x_1, \dots, x_N) = \prod_i \phi_i(x_i)^{\bar{n}_i}. \quad (4)$$

Hence, by this method we obtain a specific set of occupation numbers \bar{n}_i 's, some of which are zero. We calculated the ϕ_i 's for a lattice with 500 sites, which is effectively thermodynamic in the regime of Δ 's and U 's that we employ.

Next, we construct two more many body wave functions: $\psi_{SF}(x_1, \dots, x_N) = L^{-N/2}$ which corresponds to the superfluid state in the absence of disorder and interactions, since the condensate wave function then is $1/\sqrt{L}$, and all N particles occupy this extended state. The second wave function is $\psi_{MI}(x_1, \dots, x_N) = \prod_i e(x - x_i)$, where $e(x - x_i)$ is 1 at site i and zero otherwise. This state approaches the Mott insulator (MI) ground state for large U 's and no disorder.

In Fig. 2 we plot the total energy of each of these four many body wave functions: $\psi_{AG}, \psi_{BG}, \psi_{SF}$ and ψ_{MI} as a function of the interaction strength U , for $\Delta/J = 2$. The total energy of each of the following states is given by: $E_{BG} = \sum_i n_i E_i + U_i n_i(n_i-1)/2 + \sum_{i \neq j} U_{ij} n_i n_j$, where E_i is the eigenstate of the free single particle Hamiltonian H_0 , $E_{SF} = U_{SF} N(N-1)/2 - 2(N-1)J + Tr(H_0)$, is the single body extended state, L is the length of the lattice, $U_{SF} = g/L$, and H_0 is the free particle Hamiltonian, $E_{MI} = Tr(H_0) - 2NJ$,

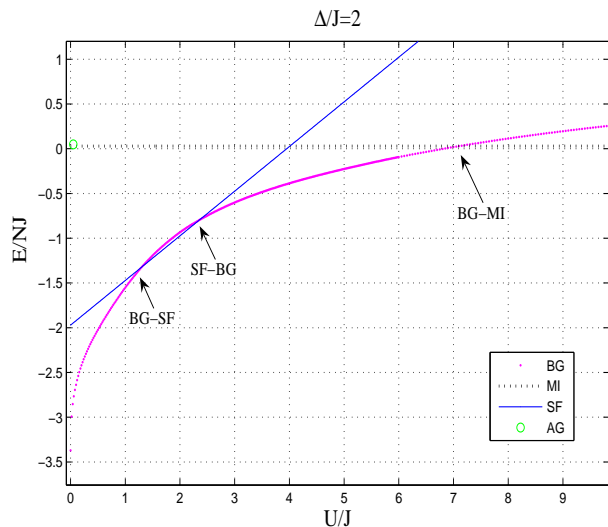


FIG. 2: Energies per particle (in units of J), of the four variational many body states: ψ_{SF} , ψ_{BG} , ψ_{MI} and ψ_{AG} , as a function of the interaction strength U/J , for $\Delta/J = 2$. The ψ_{AG} wave function has an energy which is larger than all the other states, for all non-zero Δ 's. From the graph it follows that the minimal state is ordered $\psi_{BG}, \psi_{SF}, \psi_{BG}, \psi_{MI}$, as expected. This shows that interactions tend to delocalize at small Δ 's, since they penalize localization to specific sites on the one hand, and the gain in kinetic energy of being delocalized is high, on the other hand. For larger interaction strength, this is no longer true, and a superfluid is formed. At even larger U 's the Bose glass phase is re-entered. Finally, when the interactions are even stronger, the particles will minimize the energy just localizing at the lattice sites, disregarding the disordered potential.

and $E_{AG} = NE_0 + g \int dx |\phi_0|^4 N(N-1)/2$.

We find that there are 4 regimes: for $\Delta/J < 1.3$ The minimal energy state corresponds to ψ_{BG} . For $1.3 < \Delta/J < 2.35$ we find that the minimal energy state is ψ_{SF} , which corresponds to the superfluid phase. As expected, in this regime, the condensate fraction is large, and the condensate wave function is extended. For $2.35 < \Delta/J < 7.1$ the ψ_{BG} phase is again minimal. Finally, for $\Delta/J > 7.1$ the state ψ_{MI} has minimal energy. Comparing these approximate results to the known phase diagram we conclude that this simple variational argument already exhibit the rich interplay between the kinetic energy, the disordered potential energy and the interaction energy by displaying the right ordering of the phases. Starting from the Bose glass phase and increasing the interactions, the superfluid phase can become favorable. For larger interaction strength, the Bose glass phase once again stabilizes, and for even larger interactions, finally, the Mott insulator phase wins. This correctly depicts the true behavior of the ground state [12]. Also, we note that the localized condensate is unstable for any non-zero interaction strength, hence, the Anderson glass in practice, is a phase which strictly speaking can only exist on the $U = 0$ line. Indeed, when interactions are

present, the energy cost of concentrating all the particles inside the localization length, grows without bounds in the thermodynamic limit.

We note that the one body density function ρ derived from ψ_{SF} has one macroscopic eigenfunction, but since $Tr(\rho) = N$, it follows that some of the eigenstates are zero (since the size of ρ is N by N), this is also true when the macroscopic occupation of the extended state is not total due to quantum depletion. The structure of ρ for the ψ_{BG} state is different, since instead of having long range order, the density matrix is composed of small blocks, separated by lines of zeros, indicating that there are lattice sites which are not occupied, due to their large potential energy. In both cases, the determinant of the density matrix is zero. Deep in the Mott insulator phase, the determinant is one, since the one body density function is the identity matrix. Near the transition, still in the Mott insulator phase, the ground state wave function is approximately $|1, 1, \dots, 1\rangle + (J/U) \sum_i \epsilon_i \hat{a}_{i+1}^\dagger \hat{a}_i |1, 1, \dots, 1\rangle$ which also has $\det(\rho) > 0$, $\epsilon_i \leq 1$ and are equal only when $\Delta = 0$. We therefore assert that $\det(\rho)$ is non zero in all the MI phase. This observation is just another way to express the fact that only in the Mott insulator phase, a gap is formed. When the number of occupied eigenstates (fragments) of ρ equals the number of particles, this gap will be proportional to U , which is just the cost of moving a particle to an occupied fragment.

III. DEFINITION OF CONDENSATE AND SUPERFLUID FRACTIONS

Following [22], we define the condensate fraction as the properly normalized largest eigenvalue of the one body density matrix $\rho(x, x')$. For example, when a single particle wave function $\chi(x)$ exists such that $\rho(x, x') = N\chi(x')^* \chi(x)$, then the condensate fraction is N - the number of particles in the system. Note that this definition does not imply any specific characterization of the condensate wave function $\chi(x)$, which can be extended or localized.

Superfluidity is manifest whenever there is a difference in the momentum density response to a transverse vs. a longitudinal imposed velocity field [23, 24, 25]. Here, we employ the phase twist definition [18, 26], where we impose a phase twist θ along one of the dimensions of the system, corresponding to the application of a transverse velocity field [24]. We equate the superfluid velocity v_s with $\frac{\hbar}{m} \frac{\theta}{L}$, where m is the effective mass of the atom, and L is the length of the system [24]. If the phase twist is small, then the difference in the ground state energy between the twisted (E_θ) and untwisted (E_0) systems can be attributed to the kinetic energy of the superfluid. The superfluid fraction f_s is therefore given by $(1/2)mv_s^2 f_s N = E_\theta - E_0$, with N the total number of atoms. For a normal fluid, this difference is zero, indicating an identical response to longitudinal and transverse imposed velocity fields.

IV. STUDYING THE EXACT SOLUTION

Using an exact solution for 8 particles in 8 sites, we show that with this criterion we can roughly obtain (up to our finite size resolution) the correct phase diagram of the system.

A. A lattice with one defect

The effect of a random potential on an interacting bosonic system can be identified by measuring the superfluid fraction. In practice, for a finite size system with a longitudinal extension L_z , residual superfluidity is expected to persist as long as $\xi_{loc} > L_z$. Remembering that in 1D any disorder is localizing, we first consider the simplest kind of disorder, namely, a single site defect, by adding a negative external potential at one particular site, with strength Δ . In Fig. 3, we show the result of an exact diagonalization of Eq. (1) we performed, for a 1D system with 8 atoms and 8 sites, and a varying defect strength Δ (in units of J). From the figure it is apparent that starting from the superfluid regime ($U/J = 1$) and increasing Δ , the superfluid response decreases rapidly to zero, while the condensate fraction increases to 100%. As we lower the defect, atoms localize there, without the

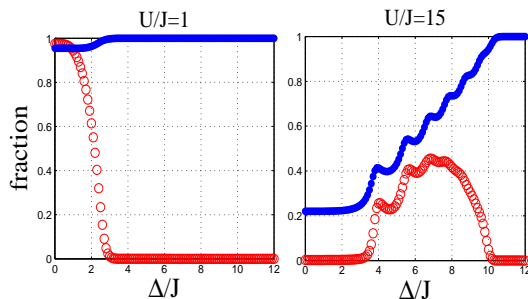


FIG. 3: Exact calculation of the largest eigenstate of ρ (filled circles) and the superfluid fraction (hollow circles) for a one dimensional lattice with 8 atoms and 8 lattice sites, and a single site defect potential $-\Delta$. (a) In the weakly interacting case ($U/J = 1$), increasing Δ/J causes all the atoms to enter the defect, so the maximal eigenvalue of the one body density is 100% of the particles, while the superfluid fraction decrease rapidly to zero, as the condensate wave function becomes localized. (b) Interactions are stronger ($U/J = 15$), so at first, lowering the defect causes delocalization of the atoms, hence, superfluidity becomes possible. Only at higher values of Δ/J it is energetically favorable to become localized at the defect, and superfluidity disappears.

ability to participate in the superfluid flow. When all the atoms are captured by the defect, they all occupy the same single particle wave function which is localized at the defect site. Starting, on the other hand, from the Mott insulator phase ($U/J = 15$), and no disorder ($\Delta = 0$), essentially all the atoms are pinned to the lattice sites. Increasing Δ causes an onset of the super-

fluid fraction, since in the presence of large interactions atoms can lower their energy by being in an extended state, hence making superfluidity possible. This will be the basis for understanding the more elaborate statement that one can onset superfluidity by increasing the disorder strength - disorder induced order. Only for larger Δ , the potential energy can compensate for the gain in interaction and kinetic energies by localizing again at the defect site, hence losing superfluidity as indicated in the figure for $\Delta/J > 10$. We confirm that the condensate wave function is indeed extended (comparable non-zero values throughout the lattice) whenever superfluidity is present [21, 27].

In this section we referred to the condensate fraction, since for a single impurity which is properly scaled it can remain finite even at the thermodynamic limit, whereas (as will be shown in the next section) for a disordered potential the condensate fraction is non zero only for a small (mesoscopic) system, and tends to zero in the thermodynamic limit.

B. Exact solution with a random on-site potential

Next, we return to the disordered potential, which acts on all 8 sites. We exactly diagonalize Eq. (1) and calculate the determinant of the one body density matrix and the superfluid fraction, for different values of U/J and Δ/J , where now, again, Δ is the width of the flat distribution from which we draw the random values of the on-site potential. The results are shown in Fig. 5. Looking at the superfluid fraction as a function of Δ for $U = 0$, we see that there is a rapid decrease of superfluidity towards zero due to the strong localization (the superfluidity starts at a non-zero value due to the finite size of our lattice). Looking at the dependence on U for $\Delta = 0$, on the other hand, we observe the usual superfluid to Mott-insulator transition, which is slightly smeared due to finite size. It is also evident from Fig. 5 that $\det \rho$ is non zero only at the Mott insulator phase.

There are two somewhat surprising features apparent in Fig. 5. First, starting from small Δ/J , and increasing the interactions, we observe the onset of superfluidity, indicating that in this regime, interactions tend to delocalize. Second, starting from the Mott-insulator phase (e.g at $U/J = 8$), and increasing the disorder strength, induces superfluidity, a phenomenon called *disorder induced order* [13].

By inspecting the eigenvalues of ρ we verified that in the regimes where the superfluid fraction f_s is zero, the condensate is fragmented, i.e. the number of non-zero eigenvalues of ρ is larger than one. While both the delocalizing role of U and disorder induced order are already present in the single defect model discussed above (see e.g. Fig. 3), fragmentation can not occur for single site “disorder” since there is always an overlap between any extended state and a localized state.

Comparing these results to the known phase diagram

[12], we find that, whenever the Bose glass phase is present, at least one but less than I eigenvalues of ρ (I being the total number of sites), are non-zero while the superfluid fraction is zero. In the Mott insulator, we observe complete fragmentation, and all the eigenvalues are non-zero (with $f_s = 0$). Finally, in the superfluid phase, some of the eigenvalues are zero, while the large eigenvalue is large (of the order of 1).

Further insight is gained by looking at specific eigenfunctions of ρ . To avoid the limitations of finite size, we inspect the Bose glass and the Mott insulator phases in a regime in the $\Delta/J - U/J$ plane, where the localization length is $\lesssim 2a$ which is less than the length of the system $I = 8a$, where a is the lattice spacing. In Fig. 4 we plot the eigenfunctions of ρ for the Mott insulator phase (MI), The superfluid phase, and the Bose glass phase. We note that In the superfluid phase ($U/J = \Delta/J = 1$), the largest eigenvalue of ρ corresponds to 95% of the particles, and the accompanying eigenstate is extended (comparable non zero values anywhere on the lattice). The last eigenvalue of ρ are zero i.e. they are unoccupied. In the Mott insulator state ($U/J = 15, \Delta/J = 1$), we see that all the eigenvalues are non-zero (the largest eigenvalue equals 22% of the total number of particles, the second largest is 17%, and the smallest is 7%), indicating complete fragmentation. Also note that the eigenfunctions are all extended. Finally, In the Bose glass regime, all the eigenfunctions are localized and some of the eigenvalues of ρ are zero, indicating partial fragmentation. In particular, due to the finite size, when the disorder is strong and the interactions are weak ($\Delta/J = 15, U/J = 1$), we find that the largest eigenvalue is 97% and the eigenfunction is localized, with $\xi \sim 2a$. When both the interactions and the disorder are strong ($U/J = 15, \Delta/J = 60$), partial fragmentation occurs. The largest eigenvalues of ρ is then 58%, the second, third and fourth largest are respectively: 17%, 13% and 12% and the rest are zero. The results presented here are of a single realization of the disorder, which we choose as representative from a larger set of disorder realizations, all with similar quantitative results.

V. A NEW ORDER PARAMETER FOR THE MOTT INSULATOR PHASE

A single, base invariant, order parameter that is non-zero only in the Mott insulator phase is $\det(\rho)$. Since $\text{Tr}(\rho) = N$, it follows that only when there is exactly one particle in each eigenfunction, all the eigenvalues of ρ are equal to 1 then $\det(\rho) = 1$, which happens in the limit $U \rightarrow \infty$. In the superfluid phase and in the Bose glass phase, $\det(\rho) = 0$. We therefore propose that the combination of the superfluid fraction and $\det(\rho)$ can distinguish the three different phases.

Inspecting $\det(\rho)$ as a function of U/J for $\Delta = 0$, we find that the superfluid to Mott insulator transition occurs at 4.6 in accordance with [8]. We also verified for

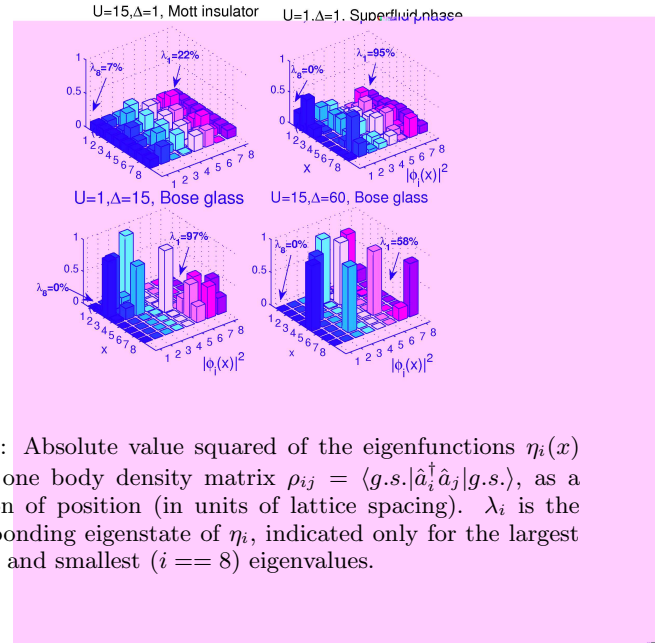


FIG. 4: Absolute value squared of the eigenfunctions $\eta_i(x)$ of the one body density matrix $\rho_{ij} = \langle g.s. | \hat{a}_i^\dagger \hat{a}_j | g.s. \rangle$, as a function of position (in units of lattice spacing). λ_i is the corresponding eigenstate of η_i , indicated only for the largest ($i = 1$) and smallest ($i = 8$) eigenvalues.

large U $\det(\rho)$ tends to unity (for example at $U = 1000$, $\det(\rho) > 0.9999$).

Thus, the Bose glass phase can be thought of as a phase composed of localized eigenstates of ρ , without phase relations between them. The total many body wave function is therefore, to leading order, $\psi(x_1, \dots, x_n) \approx \prod_i \phi_i(x_i)^{n_i}$, where $\sum n_i = N$ and some of the n_i 's are larger than 1.

In the presence of interactions, a single localized condensate at the lowest point in the potential is unstable and will tend to be fragmented to microscopic fragments (for large Δ/J) or to delocalize (for small Δ/J). This is because the interaction energy is proportional to the density squared, and thus whenever interactions are present, the ground state wave function will take a form that has the lowest possible density. A single extended condensate

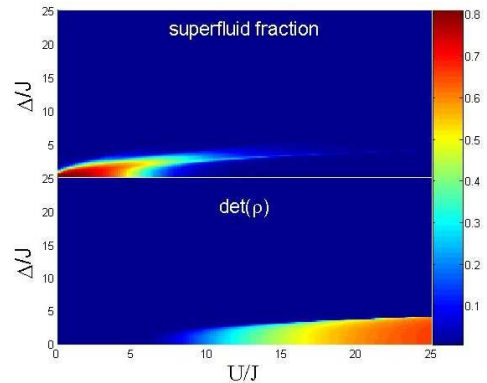


FIG. 5: Superfluid fraction, and $\det(\rho)$ as a function of U/J and Δ/J . The Bose glass phase is characterized by zero superfluidity and $\det(\rho) = 0$. The Mott insulator phase is the phase with non zero $\det \rho$ and zero superfluidity, and the superfluid phase naturally has non zero superfluidity and $\det \rho = 0$

is formed (hence invoking superfluidity), when U/J is less than the homogeneous Mott insulator-superfluid transition (so the energy penalty for occupying more than one atom per site, is small compared to the kinetic energy gain) and also Δ/J is small (so the penalty for occupying higher peaks in the disordered potential is also small). Alternatively, fragmenting to a set of non-overlapping localized condensates, each centered at a different localization center, with negligible energy differences (in the thermodynamic limit), is favorable when Δ/J is large.

Hence we expect that in the Bose glass phase, the number of fragmented condensates N_{loc} will generally be larger than one. Moreover, N_{loc} should scale like L^d where d is the dimension of the underlying lattice. In the Bose glass phase $N_{loc} < N$, since if $N_{loc} = N$, i.e. the number of fragments equals the number of sites, then an energy gap will form, whereas the Bose glass phase is not gapped. When the number of localization centers is a finite fraction of the total number of sites - a mesoscopic system, the number of atoms per localized condensate will also be a finite fraction from the total, so the condensate fraction will be non-zero. Moreover, in a finite system, we expect a regime of parameters (arbitrary disorder and weak interactions), in which the finite gap in the excitation spectrum (that exists due to the finite size) will protect a single localized condensate from fragmenting. This regime was called by several authors an Anderson glass [16, 17]. We stress that in the thermodynamic limit, the Anderson glass only exist along the line $U = 0$ (zero interactions).

Due to the limited number of lattice sites and atom number in our exact calculation, in Fig. 5, and contrary to the variational calculation presented in section two, we could not resolve the Bose glass phase intervening between the Mott insulator and the superfluid phases [1, 11, 12]. Keeping this limitation in mind, we were able to recover the correct structure of the phase diagram, as compared to previous work done with a different tool [12]. Our results can also be scaled to larger numbers. To demonstrate this, we repeated the calculations for $N = 6$ atoms and sites, and obtained a qualitatively similar phase diagram to that with $N = 8$. Moreover, using finite size scaling in 1D in the limit of $\Delta = 0$, we recover the Mott insulator transition at $U_c = 4.85J$, in good agreement with the value $4.65J$ presented in [8]. We also checked the half filling case for 6 atoms and 12 lattice sites, where since $N < I$, $\det(\rho) = 0$ indicating the absence of the MI phase as expected. Finally, we note that the fact that in the presence of disorder a Bose glass will always intervene between the Mott insulator and superfluid phases, seems more evident using our criterion, since due to the breakdown of translational invariance, the only way from the superfluid phase, where a macroscopic condensate exists (no fragmentation), to the Mott insulator state, where all the eigenfunction of ρ are occupied (complete fragmentation), goes through gradual partial fragmentation.

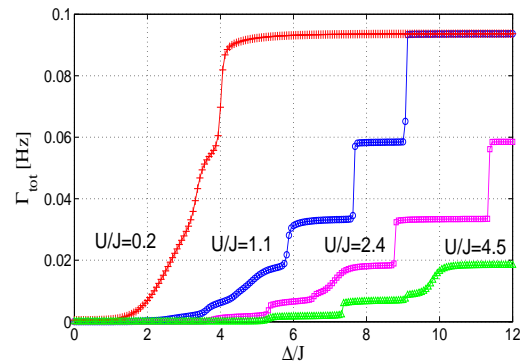


FIG. 6: Exact calculation of the total three body decay rate as a function of Δ/J for different values of the U/J (presented in units of $(U/J)_c$), for $N = I = 8$ particles and sites. Starting at the superfluid phase and increasing Δ , localization increase the density and hence the three body decay rate sharply increases. When U/J is large, the decay rate is still notably larger in the Bose glass phase compared to the Mott-insulator and superfluid phases, but it is less than the decay rate for smaller values of U , due to larger fragmentation that decreases the density. When all the particles are localized at the lowest defect, the total decay rate reaches its maximal, asymptotic value, which here is small due to small size of the system (compare with Fig. 7).

VI. USING THE THREE BODY LOSS TO IDENTIFY THE SUPERFLUID TO BOSE GLASS TRANSITION

In this section, we propose a new experimental indicator for the presence of localized states within the Bose glass phase, *when interactions are small*. To do so, we propose that for a mesoscopic system such as the systems described in [28, 29], it is possible to use the three body decay rate [30, 31], to identify the presence of localized states. We consider a collection of 1D trapped BEC's created by loading a 3D BEC into a deep two-dimensional optical lattice, and introduce a controlled disorder of strength Δ relative to J (for a more accessible experimental scheme see [29, 32]). The 1D three body loss rate coefficient is $\Gamma^{1D} = \Gamma^{3D}/12\pi^2\sigma_r^4$, where $\Gamma^{3D} = 5.810^{-42}m^6/sec$ is the corresponding 3D three body loss rate coefficient for ^{87}Rb [31], $\sigma_r = 85nm$ is the assumed radial confinement, and $L = 150\mu m$ is the condensate length [28]. Hence, we obtain that $\Gamma^{1D} = 9.410^{-16}m^2/sec$. In Fig. 6, we plot the total three body decay rate Γ_{tot} as a function of the disorder strength Δ relative to J , for different values of U/J , according to

$$\Gamma_{tot} = \sum_i \Gamma^{1D} \langle g.s. | \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \hat{a}_i | g.s. \rangle / L^3. \quad (5)$$

Looking at Fig. 7 together with Fig. 5, we observe the following features. When the system is in the Mott insulator phase, the total three body decay rate Γ_{tot} is negligible, since the amplitude for three atoms to occupy

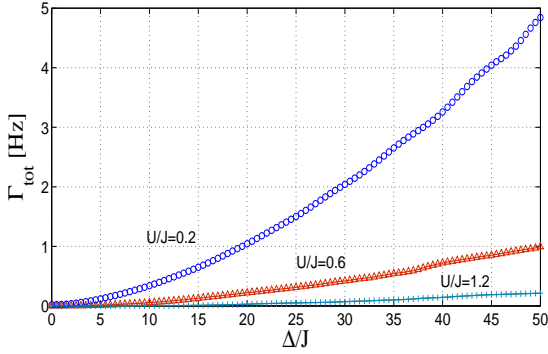


FIG. 7: Mean field calculation of the total three body decay rate as a function of Δ/J for different values of the U/J (presented in the corresponding $(U/J)_c$ units), for 101 atoms and sites. Similar to the exact result of Fig. 6, we observe that when the interactions are weak, the increase in the three body loss rate as a function of the disorder strength Δ/J is pronounced. The decay rate decreases, when the interactions increase, since the presence of interaction cause the density to decrease due to fragmentation.

the same site is effectively zero. In the superfluid phase, the bosons are now delocalized across the lattice. As a result the amplitude to occupy a single site with three or more atoms increases to a small value, and Γ_{tot} also increases to a small value. The most dramatic effect is seen when we enter the Bose glass phase and the interactions are weak. In this regime, due to the existence of localized condensates with increased density, we observe a notable increase in Γ_{tot} . When the number of fragmented condensates increases, we see a corresponding decrease in the loss, as expected. The step-like behavior of Γ_{tot} is due to the fact that the capture of another particle is a notable event, since the total number of particles is small.

In Fig. 7 we repeat the analysis, using a novel mean field simulation based on the Gutzwiller approximation [33, 34], that we developed, which is capable of working at a constant filling factor for arbitrary external potentials. With this simulation, we also calculated an upper bound for the superfluid density, and recovered qualitatively the upper part of Fig. 5. In Fig. 7, we calculated the total decay rate from a 1D lattice with 101 atoms and sites. We calculated the density profile in the presence of disorder, and the approximate ground state of the system, which we insert to Eq. 5 to obtain the loss rate. We note that although the simple Gutzwiller ansatz is unable, by construction, to recreate fragmentation, the density profile is expected to be accurate for our purposes. The results, shown in Fig. 7, are consistent with

the exact results, up to a trivial scaling of U/J and Δ/J due to the change in the critical point in the mean field model. They support our claim that three body loss is a good measure for localization in a mesoscopic system, in the regime where interactions are weak.

Finally, we note that localized condensates may also be identified by careful examination of their momentum distribution observed in time-of-flight images [8]. This can be considered as a complementary measure, to the three body loss, since as interactions increase, fragmentation becomes larger hence the loss rate decreases, while the interference is washed out.

VII. CONCLUSION AND OUTLOOK

In this paper we propose a new criterion for identifying the Bose glass phase, as a phase composed of localized fragmented eigenfunctions of ρ (with some of the localized eigenfunctions remaining unoccupied), and zero superfluidity. Starting from the Bose glass phase and increasing the interaction strength can either induce a transition to the superfluid phase, where there is one macroscopically occupied *extended* condensate, or to induce further fragmentation. When the number of fragments equals the number of sites, we enter the Mott insulator phase, which also has zero superfluidity, but with a non-zero energy gap. A summary of these observation is encapsulated by the second order parameter $\det \rho$ that we propose, alongside the superfluid fraction f_s , which is non zero only in the Mott insulator phase. We showed that $\det(\rho)$ is non zero in the Mott insulator side, both near and far from the transition to the Bose glass phase. We also argued that we can capture some of the features of the transition, in particular, that the Bose glass phase is reentered as U is increased, before entering the Mott insulator phase, by using a variational ansatz, which also obeys our criterion. We find for an exact solution for a small system with 8 particles and sites, that using our suggested criterion we can obtain the correct phase diagram (up to our limited finite size resolution), and that fragmentation mechanism occurs for large U 's.

Finally, we discuss the possibility to observe localization in a realistic experimental setup in the regime of small interaction strength, using the increase in the three body decay rate. Several intriguing effects resulting from the interplay between disorder and interactions such as disorder induced order, arise already in a single defect scheme, which may be more accessible experimentally.

This work was supported in part by the Israel Science Foundation, and by the Minerva Foundation.

[1] G. G. Matthew P. A. Fisher, Peter B. Weichman and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[2] S. A. Khairallah and D. M. Ceperley, Phys. Rev. Lett. **95**, 185301 (2005).

- [3] J. T. David R. Tilley, *Superfluidity and superconductivity* (Bristol, Adam Hilger, 1990), ISBN 0750300337.
- [4] M. Greiner, O. Mandel, T. Esslinger, T. Hansch, and I. Bloch, *Nature* **415**, 39 (2002).
- [5] S. Sachdev, *Quantum phase transitions* (Cambridge University Press, 1999), ISBN 0521582547.
- [6] G. G. Batrouni, , and R. T. Scalettar, *Phys. Rev. B.* **46**, 9051 (1992).
- [7] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, *Physical Review A* **66**, 023603 (2002).
- [8] R. Roth and K. Burnett, *Phys. Rev. A.* **68**, 023604 (2003).
- [9] I. F. Herbut, *Phys. Rev. Lett.* **79**, 3502 (1997).
- [10] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, *Phys. Rev. Lett* **91**, 080403 (2003).
- [11] B. V. Svistunov, *Phys. Rev. B.* **54**, 16131 (1996).
- [12] U. S. S. Rapsch and W. Zwerger, *Europhys. Lett* **46**, 559 (1999).
- [13] K. Hida, *Journal of the Physical Society of Japan* **72**, 688 (2003).
- [14] G. A. C. Sen and E. Dagotto, *Phys. Rev. B* **70**, 064428 (2004).
- [15] D. Pines and P. Nozieres, *Theory of Quantum Liquids*, vol. 2 (Perseus Publishing, 1999).
- [16] R. T. Scalettar, G. G. Batrouni, and G. T. Zimanyi, *Phys. Rev. Lett.* **66**, 3144 (1991).
- [17] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, *Phys. Rev. Lett.* **91**, 080403 (2003).
- [18] R. Roth and K. Burnett, *Phys. Rev. A.* **67**, 031602(R) (2003).
- [19] M. Schreiber and H. Grussbach., *Phys. Rev. Lett.* **76**, 1687 (1996).
- [20] D. S. Petrov, G. Shlyapnikov, and J. T. M. Walraven, *Phys. Rev. Lett.* **85**, 3745 (2005).
- [21] This approximation works best for weak interactions, since in this regime the noninteracting basis is still appropriate.
- [22] A. J. Leggett, *Rev. Mod. Phys.* **73**, 307 (2001).
- [23] A. J. Leggett, *Rev. Mod. Phys.* **71**, S318 (1999).
- [24] G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory*, edited by R. Clark and G. Derrick (Oliver and Boyd, 1967).
- [25] K. Huang, in *Bose-Einstein Condensation*, edited by D. s. A. Griffin and S. Stringari (Cambridge University Press, 1995).
- [26] M. E. Fisher, M. N. Barber, and D. Jasnow, *Phys. Rev. A.* **8** (1973).
- [27] E. Talbot and A. Griffin, *Phys. Rev. B.* **29**, 3952 (1984).
- [28] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Folling, I. Cirac, G. V. Shlyapnikov, T. W. Hansch, and I. Bloch, *Nature* **429**, 277 (2004).
- [29] L. Fallani, J. E. Lye, V. Guarrera, C. Fort, and M. Inguscio, *cond-mat/0603655* (2006).
- [30] E. A. Burt, R. W. Ghrist, C. J. Myatt, M. J. Holland, E. A. Cornell, and C. E. Wieman, *Phys. Rev. Lett.* **79**, 337 (1997).
- [31] J. Soding, D. Guery, Odelin, P. Desbiolles, F. Chevy, H. Inamori, and J. Dalibard, *Applied Physics B Lasers and Optics* **69**, 257 (1999).
- [32] N. Bar-Gill, R. Pugatch, E. Rowen, N. Katz, and N. Davidson, *cond-mat/0603513* (2006).
- [33] D. B. M. Dickerscheid, D. van Oosten, P. J. H. Denteneer, and H. T. C. Stoof, *Phys. Rev. A.* **68**, 043623 (2003).
- [34] B. K. Daniel S. Rokhsar, *Phys. Rev. B.* **44**, 10328 (1991).